ANNIHILATORS OF PERMUTATION MODULES

STEPHEN DOTY AND KATHRYN NYMAN

ABSTRACT. Permutation modules are fundamental in the representation theory of symmetric groups $\mathfrak{S}_n$ and their corresponding Iwahori–Hecke algebras $\mathscr{H} = \mathscr{H}(\mathfrak{S}_n)$. We find an explicit combinatorial basis for the annihilator of a permutation module in the "integral" case — showing that it is a cell ideal in G.E. Murphy's cell structure of $\mathscr{H}$. The same result holds whenever $\mathscr{H}$ is semisimple, but may fail in the non-semisimple case.

1. INTRODUCTION

Let R be a commutative ring with 1 and fix an invertible element $q \in R$. The representation theory of symmetric groups $\mathfrak{S}_n$ and the corresponding Iwahori–Hecke algebras $\mathscr{H}_{R,q} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ starts with the transitive permutation modules M_R^λ indexed by partitions λ. There are also twisted versions $\widetilde{M}_R^\lambda$ and the theory may equivalently be approached through the $\widetilde{M}_R^\lambda$ instead of the M_R^λ. The purpose of this paper is to give explicit combinatorial bases for the annihilators of M_R^λ and $\widetilde{M}_R^\lambda$ in two cases:

(i) the coefficient ring is the ring $R = \mathbb{Z}[v, v^{-1}]$ of "Laurent polynomials," where v is an indeterminate and $q = v^2$;
(ii) the coefficient ring is a field R such that $\mathscr{H}_{R,q}$ is semisimple.

The result is essentially the same in both cases; (i) is obtained in Theorem 7.3 by a refinement of an argument of Härterich [8], and (ii) is obtained in Theorem 5.2. It turns out that in these cases the annihilator of M_R^λ is a cell ideal with respect to Murphy's cellular basis of $\mathscr{H}_{R,q}$. The result does not necessarily hold in case $\mathscr{H}_{R,q}$ is not semisimple; see Example 7.4.

In case $v = 1$ the algebra $\mathscr{H}_{R,q}$ is isomorphic to the group algebra $R\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$; so by specializing v to 1 we obtain

corresponding results on annihilators of permutation modules for symmetric groups. Specifically, we obtain a description of $\mathrm{Ann}_{R\mathfrak{S}_n} M_R^\lambda$ in case (i) $R = \mathbb{Z}$, or (ii) R is a field such that $R\mathfrak{S}_n$ is semisimple.

Theorem 5.2 is easily derived from a lemma of [8] along with properties of cellular bases. To prove Theorem 7.3 we exploit Schur–Weyl duality between $\mathscr{H}$ and the quantized enveloping algebra $\mathbf{U}(\mathfrak{gl}_n)$. This gives an "integral" embedding between certain permutation modules which seems to be new; see Lemma 6.4 for the precise statement. This result, which may be of some interest in its own right, is the key step in the proof of Theorem 7.3.

2. Symmetric groups and tableaux

We denote by $\mathfrak{S}_S$ the symmetric group consisting of all bijections of a given set S; in particular we set $\mathfrak{S}_n = \mathfrak{S}_{\{1,\dots,n\}}$. We adopt the convention that elements of $\mathfrak{S}_n$ act on the *right* of their arguments, so that compositions of permutations are read from left to right. In other words, if $\sigma, \tau \in \mathfrak{S}_n$, then $i(\sigma\tau) = (i\sigma)\tau$, for any $i \in \{1,\dots,n\}$.

We write $\lambda \vDash n$ to indicate that λ is a composition of n, meaning that λ is an infinite sequence $(\lambda_1, \lambda_2, \dots)$ of nonnegative integers such that $\sum \lambda_i = n$. The individual λ_i are the parts of λ, and the largest index ℓ such that $\lambda_j = 0$ for all $j > \ell$ is the length, or number of parts, of λ. Zeros at the end of λ are usually omitted. Any composition λ may be sorted into a unique partition λ^+, in which the parts are in non-strict descending order. We write $\lambda \vdash n$ to indicate that λ is a partition of n. When $\lambda \vdash n$, we denote by λ' the transposed partition, whose Young diagram is obtained from the Young diagram of λ by writing its rows as columns.

Recall that compositions are partially ordered by dominance: given $\lambda, \mu \vDash n$, one writes $\lambda \trianglerighteq \mu$ (λ dominates μ) if $\sum_{i \leqslant j} \lambda_i \geqslant \sum_{i \leqslant j} \mu_i$ for all j, and one writes $\lambda \triangleright \mu$ (λ strictly dominates μ) if $\lambda \trianglerighteq \mu$ and the inequality $\sum_{i \leqslant j} \lambda_i \geqslant \sum_{i \leqslant j} \mu_i$ is strict for at least one j. The notations $\mu \trianglelefteq \lambda$, $\mu \triangleleft \lambda$ are respectively equivalent to $\lambda \trianglerighteq \mu$, $\lambda \triangleright \mu$. Furthermore, we note that the dominance relation reverses when taking transposes:

$$(1) \qquad\qquad \lambda' \trianglelefteq \mu' \quad \text{if and only if} \quad \lambda \trianglerighteq \mu$$

for $\lambda, \mu \vdash n$.

Let $\lambda \vDash n$. A λ-tableau t is a numbering of the boxes in the Young diagram of λ by the numbers $1, \dots, n$ such that each number appears just once. One says that t is *row-standard* if the numbers in each row are increasing read from left to right, and *standard* if t is row-standard and the numbers in each column are increasing read from top to bottom.

The group $\mathfrak{S}_n$ acts naturally on the set of tableaux, on the right, as permutations of the numbering.

3. MURPHY'S BASES OF THE HECKE ALGEBRA

Recall the definition of $\mathfrak{S}_n$ as a Coxeter group: it is the group given by the generators $s_1, \ldots, s_{n-1}$ subject to the relations

(S1) $s_i^2 = 1$;

(S2) $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$;

(S3) $s_i s_j = s_j s_i$ if $|i - j| > 1$.

We may (and will) identify the generator s_i with the transposition interchanging i, $i+1$ and fixing all other elements of $\{1, \ldots, n\}$. Every element of $\mathfrak{S}_n$ is expressible (in many ways) as a product of the form $w = s_{i_1} \cdots s_{i_k}$. Let $\ell(w)$, the length of w, be the minimum value of k in all such expressions. Any expression of the form $w = s_{i_1} \cdots s_{i_k}$ in which $k = \ell(w)$ is *reduced*.

Let R be any commutative ring with 1, and fix an invertible element q of R. The Iwahori–Hecke algebra, $\mathscr{H}_{R,q} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the associative R-algebra with 1 given by generators $T_1, T_2, \ldots, T_{n-1}$ satisfying the relations

(H1) $(T_i + 1)(T_i - q) = 0$;

(H2) $T_i T_j T_i = T_j T_i T_j$ if $|i - j| = 1$;

(H3) $T_i T_j = T_j T_i$ if $|i - j| > 1$.

In order to simplify notation, we shall henceforth write $\mathscr{H}_R$ instead of $\mathscr{H}_{R,q}$, letting q be understood.

Remark 3.1. Some authors use a different, but equivalent, quadratic relation in place of (H1): $(T_i - v)(T_i + v^{-1}) = 0$. Setting $v = q^{\frac{1}{2}}$ and extending the scalars to include $q^{\frac{1}{2}}$, one can see that this version is isomorphic to the one defined above. The results of this paper hold for the alternative version of $\mathscr{H}_R$, although many of the specific formulas are somewhat different.

If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w \in \mathfrak{S}_n$, one defines

$$T_w = T_{i_1} \cdots T_{i_k}.$$

In particular, $T_{id} = 1$ and $T_{s_i} = T_i$. The element T_w is well defined independently of the choice of reduced expression for w and satisfies

(2) $T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) > \ell(w); \\ qT_{ws_i} + (q-1)T_w & \text{if } \ell(ws_i) < \ell(w). \end{cases}$

The elements T_w, for $w \in \mathfrak{S}_n$, form an R-basis of $\mathscr{H}_R$; in particular $\mathscr{H}_R$ is free as an R-module. The generator $T_i = T_{s_i}$ is invertible in $\mathscr{H}_R$, with

$$(3) \qquad\qquad T_i^{-1} = q^{-1}(T_i - q + 1)$$

for all $i = 1, \ldots, n-1$. It follows that any T_w is invertible, with

$$(4) \qquad\qquad T_w^{-1} = T_{i_k}^{-1} \cdots T_{i_1}^{-1}$$

where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w.

We refer the reader to [13, Lemma 2.3] for a proof of the following well known result.

Lemma 3.2. *Suppose R is an integral domain. Let $*$, $\dagger$, $\sharp$ be the maps $\mathscr{H}_R \to \mathscr{H}_R$ defined on basis elements by the rules*

$$* : T_w \to T_{w^{-1}};$$
$$\dagger : T_w \to (-q)^{\ell(w)} T_w^{-1};$$
$$\sharp : T_w \to (-q)^{\ell(w)} (T_{w^{-1}})^{-1}$$

for $w \in \mathfrak{S}_n$, extended to $\mathscr{H}_R$ by linearity. Then $*$ and $\dagger$ are commuting anti-involutions of R-algebras. The map $\sharp$ is the composite of $*$ and $\dagger$; thus $\sharp$ is an involution of R-algebras.

We denote the image of any $h \in \mathscr{H}_R$ under the maps $*$, $\dagger$, and $\sharp$ by h^*, $h^\dagger$, and $h^\sharp$ respectively.

We now recall some results of Murphy [12, 13], which provide two cellular bases of $\mathscr{H}_R$. Let $\lambda \vDash n$, and let t^λ be the tableau of shape λ in which the numbers $1, \ldots, n$ have been inserted in the boxes in order from left to right along the rows. Let $\mathfrak{S}_\lambda$ be the row stabilizer of t^λ. Then

$$(5) \qquad\qquad \mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots$$

is a Young subgroup of $\mathfrak{S}_n$. Define elements

$$(6) \qquad\qquad x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w; \quad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-\ell(w)} T_w.$$

Given a tableau t of shape λ, for $\lambda \vDash n$, let $d(t)$ be the unique element of $\mathfrak{S}_n$ such that $t = t^\lambda d(t)$. Given any pair s, t of row-standard λ-tableaux, following Murphy we set

$$(7) \qquad\qquad x_{st} = T_{d(s)}^* x_\lambda T_{d(t)}, \quad y_{st} = T_{d(s)}^* y_\lambda T_{d(t)}.$$

Since x_λ and y_λ are invariant under $*$, it follows that

$$(8) \qquad\qquad x_{st}^* = x_{ts}; \quad y_{st}^* = y_{ts}$$

for any pair s, t of row-standard λ-tableaux. For any $\lambda \vdash n$ let $\mathrm{Tab}(\lambda)$ be the set of standard λ-tableaux, and set

$$\mathscr{H}_R[\rhd\lambda] = \sum_{a,b\in\mathrm{Tab}(\mu),\,\mu\rhd\lambda} Rx_{ab}; \quad \mathscr{H}_R[\unrhd\lambda] = \sum_{a,b\in\mathrm{Tab}(\mu),\,\mu\unrhd\lambda} Rx_{ab}.$$

It follows from the first equality in (8) and part (b) of the next result that $\mathscr{H}_R[\rhd\lambda]$ and $\mathscr{H}_R[\unrhd\lambda]$ are both two-sided ideals of $\mathscr{H}_R$.

Theorem 3.3. [13, Theorem 4.17 and Theorem 5.1] *Assume that R is an integral domain.*
(a) The set $\{x_{st} : s, t \in \mathrm{Tab}(\lambda), \lambda \vdash n\}$ is an R-basis of $\mathscr{H}_R$.
(b) If $\lambda \vdash n$, for any $h \in \mathscr{H}_R$, $s, t \in \mathrm{Tab}(\lambda)$ we have

$$x_{st}h = \sum_{u\in\mathrm{Tab}(\lambda)} r_h(t, u)x_{su} \quad (\mathrm{mod}\ \mathscr{H}_R[\rhd\lambda])$$

where $r_h(t, u) \in R$ is independent of s.

Note that by applying $*$ the equality in the theorem may be written in the equivalent form

$$h^*x_{ts} = \sum_{u\in\mathrm{Tab}(\lambda)} r_h(t, u)x_{us} \quad (\mathrm{mod}\ \mathscr{H}_R[\rhd\lambda]).$$

This is used, for instance, in proving that $\mathscr{H}_R[\rhd\lambda]$ and $\mathscr{H}_R[\unrhd\lambda]$ are two-sided ideals of $\mathscr{H}_R$.

Remark 3.4.
(a) The basis in part (a) of the theorem is a cellular basis in the sense of [6]. This follows from the first equality in (8) and part (b) of the theorem.
(b) For the sake of completeness, we mention that the theorem remains true if one replaces x_{st} by y_{st} throughout. Thus

$$\{y_{st} : s, t \in \mathrm{Tab}(\lambda), \lambda \vdash n\}$$

is another cellular basis of $\mathscr{H}_R$. We will not need this fact in the paper.
(c) By applying the involution $\sharp$ to any cellular basis of $\mathscr{H}_R$, we obtain another cellular basis. Thus

$$\{x_{st}^{\sharp} : s, t \in \mathrm{Tab}(\lambda), \lambda \vdash n\}, \quad \{y_{st}^{\sharp} : s, t \in \mathrm{Tab}(\lambda), \lambda \vdash n\}$$

are both cellular bases of $\mathscr{H}_R$.

For $\lambda \vdash n$, following [6] we let $C_R(\lambda)$ be the free R-module with basis $\{c_t : t \in \mathrm{Tab}(\lambda)\}$. We define an action of $\mathscr{H}_R$ on $C_R(\lambda)$ by the rule

$$(9) \qquad\qquad c_t h = \sum_{u\in\mathrm{Tab}(\lambda)} r_h(t, u)c_u$$

for any $t \in \mathrm{Tab}(\lambda)$. The element $r_h(t, u) \in R$ is determined as in part (b) of Theorem 3.3. The $\mathscr{H}_R$-modules $C_R(\lambda)$ are known as *cell modules* in the terminology of [6]. Note that for any given fixed $s \in \mathrm{Tab}(\lambda)$, the right $\mathscr{H}_R$-module $C_R(\lambda)$ is isomorphic with the submodule of $\mathscr{H}_R[\unrhd\lambda]/\mathscr{H}_R[\rhd\lambda]$ spanned by the set of all right cosets of the form

$$x_{st} + \mathscr{H}_R[\rhd\lambda] \qquad (t \in \mathrm{Tab}(\lambda)).$$

Murphy has identified the cell modules for $\mathscr{H}_R$ with respect to the cellular basis of Theorem 3.3. In [13, Theorem 5.3] he proves:

(10) $$C_R(\lambda) \simeq S_{\lambda, R} \quad \text{for any } \lambda \vdash n.$$

Here $S_{\lambda, R}$ is the linear dual $\mathrm{Hom}_R(S_R^\lambda, R)$ of the Specht module S_R^λ.

The module S_R^λ, which is a q-analogue of the corresponding Specht module for the symmetric group $\mathfrak{S}_n$, was introduced in [1, §4]. Although we need only generic properties of Specht modules associated to their interpretation as cell modules, some readers may prefer an explicit construction. One approach is to define S_R^λ as the following right ideal of $\mathscr{H}_R$:

(11) $$S_R^\lambda := x_\lambda T_{w_\lambda} y_{\lambda'} \mathscr{H}_R,$$

where the element w_λ is the unique element of $\mathfrak{S}_n$ such that $t^\lambda w_\lambda = t_\lambda$. Here t_λ is the tableau of shape λ in which the numbers $1, \ldots, n$ have been inserted in the boxes in order from top to bottom in the columns.

4. Permutation modules

In [1], [2], Dipper and James studied the right ideals $M_R^\lambda := x_\lambda \mathscr{H}_R$, for $\lambda \vDash n$, noting that they are q-analogues of the classical permutation modules (see [9]) for symmetric groups. They also studied the right ideals $\widetilde{M}_R^\lambda := y_\lambda \mathscr{H}_R$; these are q-analogues of the signed permutation modules for symmetric groups. We wish to study the annihilators of these $\mathscr{H}_R$-modules.

By [2, (2.1)], there exist elements r_λ, $r_\lambda' \in R$ such that $r_\lambda r_\lambda' = 1$ and

(12) $$x_\lambda^\sharp = r_\lambda y_\lambda; \quad y_\lambda^\sharp = r_\lambda' x_\lambda.$$

Given any right $\mathscr{H}_R$-module N, we obtain a new right $\mathscr{H}_R$-module $N^\sharp$ in the usual way, by letting $N^\sharp = N$ and twisting the original $\mathscr{H}_R$-action by the automorphism $\sharp$. On the other hand, if N is a right ideal of $\mathscr{H}_R$ then $N^\sharp = \{n^\sharp : n \in N\}$ is another right ideal of $\mathscr{H}_R$. Thus, in the case when N is a right ideal, the notation $N^\sharp$ has two meanings. However, the reader can easily check that the two possible interpretations lead to

isomorphic right $\mathscr{H}_R$-modules. From (12), it follows immediately that as right $\mathscr{H}_R$-modules we have an isomorphism

$$(13) \qquad (M_R^\lambda)^\sharp \simeq \widetilde{M_R^\lambda}$$

for any $\lambda \vDash n$. Thus, any description of $\operatorname{Ann}_{\mathscr{H}_R} M_R^\lambda$ will give immediately a description of $\operatorname{Ann}_{\mathscr{H}_R} \widetilde{M_R^\lambda}$, simply by applying the involution $\sharp$. So we focus on obtaining a description of the former.

Let $\mathscr{H}_R(\lambda)$ be the subalgebra of $\mathscr{H}_R$ generated by all the T_i except $T_{\lambda_1}, T_{\lambda_1+\lambda_2}, \ldots, T_{\lambda_1+\cdots+\lambda_k}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ has length k. By [11, Corollary 1.14], there are two one-dimensional representations $1_{\mathscr{H}_R}$ and $\varepsilon_{\mathscr{H}_R}$ of $\mathscr{H}_R$ defined on basis elements by

$$(14) \qquad 1_{\mathscr{H}_R}(T_w) = q^{\ell(w)}; \quad \varepsilon_{\mathscr{H}_R}(T_w) = (-1)^{\ell(w)}$$

for any $w \in \mathfrak{S}_n$. These are known as the *trivial* and *sign* representations, respectively; by abuse of notation we denote the corresponding right $\mathscr{H}_R$-modules by the same symbols. As right $\mathscr{H}_R$-modules, we have isomorphisms

$$(15) \qquad M_R^\lambda \simeq 1_{\mathscr{H}_R} \otimes_{\mathscr{H}_R(\lambda)} \mathscr{H}_R, \quad \widetilde{M_R^\lambda} \simeq \varepsilon_{\mathscr{H}_R} \otimes_{\mathscr{H}_R(\lambda)} \mathscr{H}_R.$$

thus justifying the terminology "permutation" and "signed permutation" for these modules.

Let $\lambda \vDash n$. For typographical reasons, we denote by $x_{\lambda t}$ the element x_{st} with $s = t^\lambda$. Thus $x_{\lambda t} = x_\lambda T_{d(t)}$, for any row-standard tableau t of shape λ. We will need the following result of Dipper and James, which gives a basis of M_R^λ and determines the action of $\mathscr{H}_R$ on basis elements.

Lemma 4.1. [1, Lemma 3.2] *Assume that R is an integral domain, and let $\lambda \vDash n$.*
(a) $\{x_{\lambda t} : t \text{ row-standard of shape } \lambda\}$ is an R-basis of $M_R^\lambda = x_\lambda \mathscr{H}_R$.
(b) Suppose t is row-standard of shape λ, and set $u = ts_i$. Let $\operatorname{row}_t(j)$ be the row index of j in t. Then

$$x_{\lambda t} T_i = \begin{cases} q x_{\lambda t} & \text{if } \operatorname{row}_t(i) = \operatorname{row}_t(i+1) \\ x_{\lambda u} & \text{if } \operatorname{row}_t(i) < \operatorname{row}_t(i+1) \\ q x_{\lambda u} + (q-1) x_{\lambda t} & \text{if } \operatorname{row}_t(i) > \operatorname{row}_t(i+1). \end{cases}$$

Note that for $\lambda \vDash n$, $M_R^\lambda \simeq M_R^{\lambda^+}$, where λ^+ is the unique partition in the $\mathfrak{S}_n$-orbit of λ. Thus, when considering the annihilator of M_R^λ, it is enough to restrict our attention to the case in which $\lambda \vdash n$.

5. The annihilator in the semisimple case

We will require a preparatory lemma of Härterich. To formulate it, we need to extend the dominance order on compositions to the set of row-standard tableaux, as follows. Let t be a row-standard λ-tableau, where $\lambda \vDash n$. For any $j \leqslant n$ denote by $t_{\downarrow j}$ the row-standard tableau that results from throwing away all boxes of t containing a number bigger than j. Let $[t_{\downarrow j}]$ be the corresponding composition of j (the composition defining the shape of $t_{\downarrow j}$). Given row-standard tableaux s and t with the same number n of boxes, define

$s \trianglerighteq t$ if for each $j \leqslant r$, $[s_{\downarrow j}] \trianglerighteq [t_{\downarrow j}]$;

$s \vartriangleright t$ if for each $j \leqslant r$, $[s_{\downarrow j}] \vartriangleright [t_{\downarrow j}]$.

Note that if s and t are standard tableaux, respectively of shape λ and μ, where λ and μ are partitions of n, then $s \trianglerighteq t$ if and only if $t' \trianglerighteq s'$. Here t' denotes the transposed tableau of t, obtained from t by writing its rows as columns. The dominance order on tableaux extends naturally to pairs of tableaux, by defining:

(16) $(s,t) \trianglerighteq (u,w)$ iff $s \trianglerighteq u$ and $t \trianglerighteq w$.

For a set P of partitions of n, set $\mathscr{H}_R[P] = \sum_{s,t} R x_{st}$, where the sum is taken over the set of pairs $s, t \in \mathrm{Tab}(\mu)$, for $\mu \in P$. If P is closed with respect to $\trianglerighteq$ (i.e., $\nu \vdash n$, $\mu \in P$, and $\nu \trianglerighteq \mu$ implies $\nu \in P$), then $\mathscr{H}_R[P]$ is a two sided ideal of $\mathscr{H}_R$. For $\lambda \vdash n$, the set $P = \{\mu \vdash n : \mu \ntrianglelefteq \lambda'\}$ is closed with respect to $\trianglerighteq$, so $\mathscr{H}_R[\ntrianglelefteq \lambda']$ is a two-sided ideal of $\mathscr{H}_R$. Thus

$$\mathscr{H}_R[\ntrianglelefteq \lambda']^\sharp = \{h^\sharp | h \in \mathscr{H}_R[\ntrianglelefteq \lambda']\}$$

is also a two-sided ideal of $\mathscr{H}_R$. We remark that these ideals are free as R-modules and have R-rank $\sum_{\mu' \ntrianglelefteq \lambda'} (\dim_R S_R^\mu)^2$. This follows from the cellular axioms and the various identifications made above.

The following result is a variant of [8, Lemma 3].

Lemma 5.1 (Härterich). *Let R be an integral domain. For any $\lambda \vdash n$ we have $\mathscr{H}_R[\ntrianglelefteq \lambda']^\sharp \subseteq \mathrm{Ann}_{\mathscr{H}_R} M_R^\lambda$.*

Proof. Let $a, b \in \mathrm{Tab}(\mu)$ where $\mu \ntrianglelefteq \lambda'$, and let c be any row-standard λ-tableau. We have $c \ntrianglelefteq a'$, since otherwise $\lambda = [c_{\downarrow n}] \trianglelefteq [a'_{\downarrow n}] = \mu'$. By Lemma 4.12 of [13], $x_{ab}^\dagger x_{c\lambda} = 0$. Applying $*$, we have $x_{c\lambda}^* (x_{ab}^\dagger)^* = x_{\lambda c} x_{ab}^\sharp = 0$. Now, since $\{x_{\lambda c} : c$ is a row-standard λ-tableau$\}$ is a basis of M_R^λ, we have $x_{ab}^\sharp \in \mathrm{Ann}_{\mathscr{H}_R} M_R^\lambda$ for a, b of shape $\mu \ntrianglelefteq \lambda'$. $\square$

Theorem 5.2. *Let R be a field, and suppose that $\mathscr{H}_R$ is semisimple. Then, for any $\lambda \vdash n$, equality holds in the inclusion in the preceding lemma; i.e., we have $\mathscr{H}_R[\ntrianglelefteq \lambda']^\sharp = \mathrm{Ann}_{\mathscr{H}_R} M_R^\lambda$.*

Proof. The semisimplicity of $\mathscr{H}_R$ implies that M_R^λ is completely reducible as an $\mathscr{H}_R$-module, with irreducible factors of the form S_R^μ for various $\mu \trianglerighteq \lambda$. By the q-analogue of Young's rule [13, Theorem 7.2], each S_R^μ for any μ satisfying $\mu \trianglerighteq \lambda$ occurs at least once in M_R^λ. Thus, it follows by the theory of semisimple algebras that

$$\dim_R(\mathscr{H}_R / \operatorname{Ann}_{\mathscr{H}_R} M_R^\lambda) \geqslant \sum_{\mu \trianglerighteq \lambda} (\dim_R S_R^\mu)^2,$$

or, equivalently,

$$\dim_R \operatorname{Ann}_{\mathscr{H}_R} M_R^\lambda \leqslant \sum_{\mu \ntrianglerighteq \lambda} (\dim_R S_R^\mu)^2.$$

But the remark at the end of the paragraph preceding Lemma 5.1 gives the equality

$$\sum_{\mu' \ntrianglelefteq \lambda'} (\dim_R S_R^\mu)^2 = \dim_R \mathscr{H}_R[\ntrianglelefteq \lambda']^\sharp$$

and applying the equivalence between the two conditions $\mu' \ntrianglelefteq \lambda'$ and $\mu \ntrianglerighteq \lambda$, we conclude that

$$\dim_R \operatorname{Ann}_{\mathscr{H}_R} M_R^\lambda \leqslant \dim_R \mathscr{H}_R[\ntrianglelefteq \lambda']^\sharp.$$

Finally, Lemma 5.1 gives the opposite inequality, thus proving the result. $\qquad\square$

Remark 5.3. Let R be a field. It is well known [4, Theorem 4.3] that $\mathscr{H}_R$ is semisimple unless one of the following holds:

(1) $q \neq 1$ and q is a primitive eth root of 1, where $e \leqslant n$;
(2) $q = 1$ and the characteristic of R is $\leqslant n$.

In case $\mathscr{H}_R$ is not semisimple, the inclusion of Lemma 5.1 may be strict (see Example 7.4).

6. Tensor space

Henceforth we will work over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate, with quotient field $\mathbb{Q}(v)$. We set $q = v^2$ and consider the Hecke algebra $\mathscr{H}_{\mathcal{A}}$ over the ring $\mathcal{A}$, defined by the generators and relations as in §3. For any commutative ring R with a chosen invertible element v we set $q = v^2$ in R. Then

$$(17) \qquad\qquad R \otimes_{\mathcal{A}} (\mathscr{H}_{\mathcal{A}}) \simeq \mathscr{H}_R,$$

where R is regarded as an $\mathcal{A}$-algebra by means of the ring homomorphism $\mathcal{A} \to R$ sending $v \in \mathcal{A}$ to $v \in R$. The isomorphism in (17) is determined by sending $1 \otimes T_i \to T_i$ for all $i = 1, \ldots, n-1$. Moreover, we have

$$(18) \qquad\qquad R \otimes_{\mathcal{A}} (M_{\mathcal{A}}^\lambda) \simeq M_R^\lambda$$

for any $\lambda \vDash n$, and any R. Our goal is to determine the annihilator of the integral permutation module $M_{\mathcal{A}}^{\lambda}$.

We will need the Drinfeld–Jimbo quantized enveloping algebra corresponding to the Lie algebra $\mathfrak{gl}_m$ for $m \geqslant 2$. Let Y be the free abelian group with basis $H_1, \ldots, H_m$. Let $\varepsilon_1, \ldots, \varepsilon_m \in X := Y^*$ be the corresponding dual basis; ε_i is given by $\varepsilon_i(H_j) := \delta_{i,j}$ for $j = 1, \ldots, m$. For $i = 1, \ldots, m-1$ let $\alpha_i \in X$ be given by $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. Define a partial order $\leqslant$ on X by $\lambda \leqslant \mu$ if and only if $\mu - \lambda \in \sum_i \mathbb{N}\alpha_i$.

Define an associative algebra $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_n)$ (with 1) over $\mathbb{Q}(v)$ by the generators

$$E_i, F_i \quad (i = 1, \ldots, m-1), \qquad v^h \quad (h \in Y)$$

subject to the defining relations

(U1) $v^0 = 1, \quad v^h v^{h'} = v^{h+h'}$

(U2) $v^h E_i v^{-h} = v^{\alpha_i(h)} E_i, \quad v^h F_i v^{-h} = v^{-\alpha_i(h)} F_i$

(U3) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}}$ where $K_i := v^{H_i - H_{i+1}}$

(U4) $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ if $|i - j| = 1$

(U5) $F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ if $|i - j| = 1$

(U6) $E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i$ if $|i - j| > 1$

for $1 \leqslant i, j \leqslant m-1$ and $h, h' \in Y$. The subalgebra of $\mathbf{U}$ generated by all the E_i, F_i, K_i (for $i = 1, \ldots, m-1$) is denoted by $\mathbf{U}(\mathfrak{sl}_m)$; this is the quantized enveloping algebra corresponding to the Lie algebra $\mathfrak{sl}_m$.

In the remainder of this section all tensor products will be over $\mathbb{Q}(v)$ unless specified otherwise. There exist unique algebra maps $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ (where $\mathbf{U} \otimes \mathbf{U}$ is regarded as an algebra in the usual way) and $\epsilon : \mathbf{U} \to \mathbb{Q}(v)$ such that

$$\Delta(v^h) = v^h \otimes v^h, \quad \epsilon(v^h) = 1$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \epsilon(E_i) = 0$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \epsilon(F_i) = 0$$

for any $i = 1, \ldots, m-1$, and $h \in Y$. The map Δ defines a comultiplication and the map ϵ a counit which together define a coalgebra structure on $\mathbf{U}$. (Actually, there is a well defined antipode $S : \mathbf{U} \to \mathbf{U}^{opp}$ which together with Δ, ϵ give $\mathbf{U}$ a Hopf algebra structure, but we shall not need it.) The map Δ induces a map $\Delta^{(n)} : \mathbf{U} \to \mathbf{U}^{\otimes n}$ defined as the composite

$$\mathbf{U} \xrightarrow{\Delta} \mathbf{U} \otimes \mathbf{U} \xrightarrow{\Delta \otimes 1} \mathbf{U} \otimes \mathbf{U} \otimes \mathbf{U} \xrightarrow{\Delta \otimes 1 \otimes 1} \cdots \xrightarrow{\Delta \otimes 1 \otimes \cdots \otimes 1} \mathbf{U}^{\otimes n}$$

for $n \geqslant 2$. Note that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, etc. We have $\Delta = \Delta^{(2)}$ and

$$\Delta^{(n)}(v^h) = v^h \otimes v^h \otimes \cdots \otimes v^h;$$

$$\Delta^{(n)}(E_i) = E_i \otimes K_i \otimes \cdots \otimes K_i \ + \ 1 \otimes E_i \otimes K_i \otimes \cdots \otimes K_i \ +$$
$$\cdots \ + \ 1 \otimes \cdots \otimes 1 \otimes E_i;$$

$$\Delta^{(n)}(F_i) = F_i \otimes 1 \otimes \cdots \otimes 1 \ + \ K_i^{-1} \otimes F_i \otimes 1 \otimes \cdots \otimes 1 \ +$$
$$\cdots \ + \ K_i^{-1} \otimes \cdots \otimes K_i^{-1} \otimes F_i.$$

The map $\Delta^{(n)}$ is used to put a **U**-module structure on the nth tensor power of a given **U**-module.

Set $V = \mathbb{Q}(v)^m$ with canonical basis $\{e_1, \ldots, e_m\}$. Define a left action of **U** on V by

$$(19) \qquad E_i e_j = \delta_{j,i+1} e_i; \quad F_i e_j = \delta_{j,i} e_{i+1}; \quad v^h e_j = v^{\varepsilon_j(h)} e_j$$

for $j = 1, \ldots, m$, $i = 1, \ldots, m-1$, $h \in Y$. This action makes V into a **U**-module, and by applying $\Delta^{(n)}$ one makes the tensor power $V^{\otimes n}$ into a **U**-module as well.

Let $I(m, n)$ be the set of all finite sequences $\mathbf{i} = (i_1, \ldots, i_n)$ where each $i_j \in \{1, \ldots, m\}$. Set

$$(20) \qquad\qquad e_{\mathbf{i}} := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n};$$

in terms of this notation $\{e_{\mathbf{i}} : \mathbf{i} \in I(m, n)\}$ is a basis for $V^{\otimes n}$. Let $\mathbf{U}^0$ be the subalgebra of **U** generated by the v^h for $h \in Y$. Since

$$(21) \qquad\qquad v^h e_{\mathbf{i}} = v^{\lambda(h)} e_{\mathbf{i}}$$

for all $h \in Y$, the vectors $e_{\mathbf{i}}$ are weight vectors (of weight λ) for the action of $\mathbf{U}^0$. Here $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m \in X$ where each $\lambda_k =$ the number of j such that $i_j = k$. It follows that

$$(22) \qquad\qquad V^{\otimes n} = \bigoplus_{\lambda \in X} (V^{\otimes n})_\lambda$$

where $(V^{\otimes n})_\lambda$ is the $\mathbb{Q}(v)$-span of all $e_{\mathbf{i}}$ of weight λ.

The algebra **U** has q-analogues of the standard root vectors in $\mathfrak{gl}_m$, defined as follows. For a positive root $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leqslant i < j \leqslant m$, if $j - i = 1$ then set $E_{i,j} = E_i$ and $E_{j,i} = F_i$. If $j - i > 1$, we assume by induction that $E_{i+1,j}$ and $E_{j,i+1}$ have already been defined, and set (following Jimbo [10, Proposition 1], Xi [14, Section 5.6])

$$(23) \quad E_{i,j} = v^{-1} E_i E_{i+1,j} - E_{i+1,j} E_i; \quad E_{j,i} = v E_{j,i+1} F_i - F_i E_{j,i+1}.$$

Set $K_{i,j} = v^{H_i - H_j}$ for each $1 \leqslant i < j \leqslant m$. We note the following results.

Lemma 6.1. *For* $1 \leqslant i < j \leqslant m$ *the triple* $\{E_{i,j}, K_{i,j}, E_{j,i}\}$ *generates a subalgebra of* $\mathbf{U}$ *isomorphic with* $\mathbf{U}(\mathfrak{sl}_2)$.

Proof. Set $E = E_{i,j}$, $K = K_{i,j}$, and $F = E_{j,i}$ for $i < j$. We need only show that E, K, and F satisfy the defining relations for $\mathbf{U}(\mathfrak{sl}_2)$:

(a) $KEK^{-1} = v^2 E$; $KFK^{-1} = v^{-2}F$

(b) $EF - FE = \frac{K - K^{-1}}{v - v^{-1}}$

In the case $j - i = 1$, (a) follows from (U1) and (U2) while (b) is immediate from (U3). Assume $j - i > 1$, and assume that (a) and (b) hold for $E' = E_{i+1,j}$, $K' = K_{i-1,j}$, and $F' = E_{j,i+1}$ in place of E, K, and F respectively. Then, using 23 and the equality $K = K_i K'$, we have

$$
\begin{aligned}
KEK^{-1} &= (K_i K')(v^{-1} E_i E' - E' E_i)(K_i K')^{-1} \\
&= v^{-1}(K_i E_i K_i^{-1})(K' E' K'^{-1}) - (K' E' K'^{-1})(K_i E_i K_i^{-1}) \\
&= v^2(v^{-1} E_i E' - E' E_i) \\
&= v^2 E
\end{aligned}
$$

which proves the first relation in (a). The second relation in (a) and relation (b) are proved by similar calculations. $\square$

The following result was observed in [5, Lemma 4.2] although in a slightly different context. We adapt the argument from [5] to our situation.

Lemma 6.2. *Let* $\mu \in X$, *and set* $\mu = \sum_k \mu_k \varepsilon_k$. *Assume* $\mu_k \geq 0$ *for* $k = 1, \ldots, m$. *Suppose that* $i < j$ *and* $\mu_i - \mu_j > 0$. *Then the action of* $E_{j,i}$ *gives a* $\mathbb{Q}(v)$-*linear injection* $(V^{\otimes n})_\mu \to (V^{\otimes n})_{\mu - \varepsilon_i + \varepsilon_j}$.

Proof. It is clear that the action of $E_{j,i}$ gives a linear map from $(V^{\otimes n})_\mu$ into $(V^{\otimes n})_{\mu - \varepsilon_i + \varepsilon_j}$, so we need only to prove that the map is injective. Set

$$E = E_{i,j}, \qquad K = K_{i,j}, \qquad F = E_{j,i}.$$

By induction on r, one proves easily that

$$E^r F - F E^r = \sum_{\substack{s+s'=r-1 \\ s,s' \geq 0}} E^s \frac{K - K^{-1}}{v - v^{-1}} E^{s'}$$

for any positive integer r. Fix some $0 \neq u \in (V^{\otimes n})_\mu$ and choose r so that $E^r u = 0$ but $E^s u \neq 0$ for any $s < r$. This is possible since E acts locally nilpotently on any finite-dimensional $\mathbf{U}$-module. Then

$$E^r F u = (E^r F - F E^r) u$$

$$= \sum_{\substack{s+s'=r-1 \\ s,s' \geq 0}} E^s \frac{K - K^{-1}}{v - v^{-1}} E^{s'} u.$$

But a calculation in $\mathbf{U}(\mathfrak{sl}_2)$ shows that

$$\frac{K - K^{-1}}{v - v^{-1}} E^{s'} u = v^{\mu_i - \mu_j + 2s'} E^{s'} u,$$

and it follows that $E^r F u = \left(\sum_{s'=0}^{r-1} v^{\mu_i - \mu_j + 2s'} \right) E^{r-1} u \neq 0$. This proves that $F u = E_{j,i} u \neq 0$, so $E_{j,i}$ is injective, as desired. $\qquad\square$

Consider the action of $\mathfrak{S}_n$ (on the right) on $V^{\otimes n}$ by place permutation:

$$(24) \qquad (u_1 \otimes u_2 \otimes \cdots \otimes u_n) w = u_{1w^{-1}} \otimes u_{2w^{-1}} \otimes \cdots \otimes u_{nw^{-1}}$$

for $w \in \mathfrak{S}_n$, $u_1, \ldots, u_n \in V$. Note that $\mathfrak{S}_n$ acts (on the right) on the set $I(m,n)$, by the rule

$$(25) \qquad \mathbf{i} \cdot w = (i_{1w^{-1}}.i_{2w^{-1}}, \ldots, i_{nw^{-1}})$$

for $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $w \in \mathfrak{S}_n$. With this notation, the action of $\mathfrak{S}_n$ on the basis elements $e_{\mathbf{i}}$ of $V^{\otimes n}$ is given by $e_{\mathbf{i}} w = e_{\mathbf{i} \cdot w}$.

We now define a right action of $\mathscr{H} = \mathscr{H}_{\mathbb{Q}(v)}$ on $V^{\otimes n}$ on basis elements by

$$(26) \qquad e_{\mathbf{i}} T_k = \begin{cases} v^2 e_{\mathbf{i}} & \text{if } i_k = i_{k+1} \\ v e_{\mathbf{i} \cdot s_k} & \text{if } i_k < i_{k+1} \\ v e_{\mathbf{i} \cdot s_k} + (v^2 - 1) e_{\mathbf{i}} & \text{if } i_k > i_{k+1}. \end{cases}$$

for any $\mathbf{i} \in I(m,n)$, and any $k = 1, \ldots, n-1$. Notice that when $v = 1$ this is the same as the place permutation action of $\mathfrak{S}_n$. One may check that the actions of $\mathscr{H}$ and $\mathbf{U}$ on $V^{\otimes n}$ commute; this goes back to [10]. The commutativity of the two actions means that a weight space $(V^{\otimes n})_\lambda$ is a right $\mathscr{H}$-module, for any $\lambda \in X$.

Lemma 6.3. *Let $\lambda \vDash n$, and assume $m \geqslant \ell(\lambda)$. Identify $\lambda = (\lambda_1, \ldots, \lambda_m)$ with the element $\sum_i \lambda_i \varepsilon_i$ of X. Write $M^\lambda := M^\lambda_{\mathbb{Q}(v)}$. Then $M^\lambda \simeq (V^{\otimes n})_\lambda$ as right $\mathscr{H}$-modules.*

Proof. (See [3, §2] or [7, §1].) Any $\underline{i} \in I(m,n)$ determines a unique row-standard tableau $t(\underline{i})$, in which j appears in row k whenever $i_j = k$, for $j = 1, \ldots, n$. For example,

$$t((2,1,2,3)) = \begin{array}{c} \boxed{2} \\ \boxed{1}\boxed{3} \\ \boxed{4} \end{array}$$

Let $\lambda \vDash n$ and let $\ell(\lambda) \leq m$. Define a map $\phi : (V^{\otimes n})_\lambda \to M_R^\lambda$ on a basis by

$$\phi(e_{\underline{i}}) = v^N x_{\lambda t(\underline{i})},$$

where $\underline{i} \in I(m, n)$ indexes a simple tensor $e_{\underline{i}}$ of weight λ, and N is the number of pairs $s < s'$ such that $i_s < i_{s'}$. The map ϕ gives the desired isomorphism. □

The next result does not seem to have been previously observed in the literature. By specializing v to 1 it gives in particular embeddings of permutation modules for $\mathbb{Q}\mathfrak{S}_n$.

Lemma 6.4. *Suppose that $R = \mathbb{Q}(v)$ with v an indeterminate, and $q = v^2$. If $\lambda \unrhd \mu$, then M^λ is isomorphic with an $\mathscr{H}$-submodule of M^μ.*

Proof. It suffices to prove the lemma for the case $\lambda \rhd \mu$ and λ, μ are adjacent in the dominance order. In this case, the Young diagram of λ is obtained from that of μ by raising one box from the jth to the ith row, where $i < j$. This implies the statement in general, since whenever $\lambda \rhd \mu$, one gets from μ to λ by a finite succession of such box-raising operations.

Now, for $\lambda \rhd \mu$ and λ, μ adjacent, we have $\lambda = \mu + \varepsilon_i - \varepsilon_j$, and the result follows from Lemmas 6.2 and 6.3. □

Remark 6.5. Suppose that R is a field and $v = 1$. It is easy to see that Lemma 6.4 may be false if the characteristic of R is positive. For example, consider the map $f_{12} : M^{(3,1)} \to M^{(2,2)}$. The matrix of this map, with respect to a convenient choice of ordering of bases, is as follows:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and one observes that each column has precisely two 1's, so the sum of the rows is zero in characteristic 2. Thus, in characteristic 2 the matrix has rank strictly less than 4, and thus the map is not injective. (This example is related to Example 7.4 below.)

It is easy to construct similar examples showing that Lemma 6.4 fails in any given positive characteristic, in the $v = 1$ case.

7. THE ANNIHILATOR IN THE INTEGRAL CASE

In this section we work over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. In Theorem 7.3 below, we describe $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda$, using a refinement of an argument of [8]. The following result will be crucial in the proof of Theorem 7.3.

Lemma 7.1. *If $\lambda \unrhd \mu$ for $\lambda, \mu \vdash n$, then $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda \supseteq \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\mu$.*

Proof. This is an immediate consequence of Lemma 6.4, since the restriction of an injective map is injective. □

Next, we will need a result of Murphy. Let $a, b \in \mathscr{H}_\mathcal{A}$, and let (a, b) denote the coefficient of T_1 in the expression $ab^* = \sum_{w \in \mathfrak{S}_n} c_w T_w$, where $c_w \in \mathcal{A}$. Then $(\ ,\ )$ is a non-degenerate, symmetric bilinear form on $\mathscr{H}_\mathcal{A}$. This bilinear form satisfies the properties

$$(27) \qquad (a, bd) = (ad^*, b); \quad (a, db) = (d^*a, b)$$

for any $a, b, d \in \mathscr{H}_\mathcal{A}$.

Lemma 7.2 (Murphy [13, Lemma 4.16]). *Let s and t be row-standard μ-tableaux and let $u, w \in \mathrm{Tab}(\lambda)$, where $\mu \vDash n$ and $\lambda \vdash n$. Then:*
  (a) $(x_{st}, x_{uw}^\sharp) = 0$ *unless* $(u', w') \trianglerighteq (s, t)$;
  (b) $(x_{u'w'}, x_{uw}^\sharp) = \pm v^{2b}$ *where* $b = \ell(d(t_\lambda)) + \sum_{i \geq 1} \lambda_i'(\lambda_i' - 1)/2$.

The following result is an "integral" version of Theorem 5.2.

Theorem 7.3. *For any $\lambda \vdash n$ we have* $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda = \mathscr{H}_\mathcal{A}[\ntrianglelefteq \lambda']^\sharp$.

Proof. By Lemma 5.1, we have already the containment $\mathscr{H}_\mathcal{A}[\ntrianglelefteq \lambda']^\sharp \subseteq \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda$, so we have only to prove the reverse containment. Let

$$0 \neq a = \sum_{\substack{s,t \in \mathrm{Tab}(\lambda) \\ \lambda \vdash n}} a_{st} x_{st}^\sharp \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda,$$

where $a_{st} \in \mathcal{A}$. It suffices to prove that $a_{st} = 0$ for all s and t of some shape $\mu \trianglelefteq \lambda'$.

Suppose not. By Lemma 5.1 we have $\sum_{(s,t) \in \Phi} a_{st} x_{st}^\sharp \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda$, hence it follows that

$$0 \neq a_0 = \sum_{(s,t) \in \Phi^c} a_{st} x_{st}^\sharp \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda,$$

where $\Phi = \bigsqcup_{\mu \ntrianglelefteq \lambda'} \mathrm{Tab}(\mu) \times \mathrm{Tab}(\mu)$ and $\Phi^c = \bigsqcup_{\mu \trianglelefteq \lambda'} \mathrm{Tab}(\mu) \times \mathrm{Tab}(\mu)$. Let (s_0, t_0) be a minimal pair in Φ^c such that $a_{s_0 t_0} \neq 0$; i.e.,

$$a_{st} = 0 \text{ for all } (s, t) \in \Phi^c \text{ satisfying } (s, t) \lhd (s_0, t_0).$$

Let λ_0 be the shape of s_0' (= shape of t_0'). Then λ_0' is the shape of s_0, t_0, so $\lambda_0' \trianglelefteq \lambda'$, and hence $\lambda \trianglelefteq \lambda_0$. By Lemma 7.1, anything annihilating $M_\mathcal{A}^\lambda$ also annihilates $M_\mathcal{A}^{\lambda_0}$. Thus, it follows that $x_{\lambda_0} T_{d(t_0')} a_0 = 0$. Hence, $T_{d(s_0')}^* x_{\lambda_0} T_{d(t_0')} a_0 = x_{s_0' t_0'} a_0 = 0$. So, by the definition of the bilinear

form, we have

$$0 = \left(x_{s_0' t_0'}, \sum_{(s,t) \in \Phi^c} a_{st} x_{st}^\sharp \right) = \sum_{(s,t) \in \Phi^c} a_{st} (x_{s_0' t_0'}, x_{st}^\sharp).$$

By Lemma 7.2(a), all the terms in the last sum are zero unless $(s', t') \trianglerighteq (s_0', t_0')$, i.e., unless $(s, t) \trianglelefteq (s_0, t_0)$. By the minimality assumption, $a_{st} = 0$ for all pairs (s, t) strictly less dominant than (s_0, t_0). Thus, the sum $\sum_{(s,t) \in \Phi^c} a_{st} (x_{s_0' t_0'}, x_{st}^\sharp)$ collapses to a single term $a_{s_0 t_0} (x_{s_0' t_0'}, x_{s_0 t_0}^\sharp)$, and by our assumption and Lemma 7.2(b) this term is nonzero. This is a contradiction. This contradiction establishes the desired opposite inclusion, and proves the theorem. $\qquad\square$

Example 7.4. We can now give an example to show that the annihilator of M_R^λ depends on R, even when R is a field. We take $v = 1$, $\lambda = (2, 2)$, and let R be a field of characteristic 2. It is quickly seen that the element $r = (23) + (1342) + (1243) + (14)$ (written in cycle notation) annihilates $M_R^{(2,2)}$ in characteristic 2. However, we claim r is not in the span of the basis elements of $\mathrm{Ann}_{\mathbb{Z}\mathfrak{S}_4} M_{\mathbb{Z}}^{(2,2)}$ with coefficients reduced modulo 2.

By Theorem 7.3, for standard tableaux $a = \boxed{1\,2\,3\,4}$, $b = \begin{smallmatrix}\boxed{1\,2\,3}\\ \boxed{4}\end{smallmatrix}$, $c = \begin{smallmatrix}\boxed{1\,2\,4}\\ \boxed{3}\end{smallmatrix}$, and $d = \begin{smallmatrix}\boxed{1\,3\,4}\\ \boxed{2}\end{smallmatrix}$, the following 10 Murphy elements form a basis for $\mathrm{Ann}_{\mathbb{Z}\mathfrak{S}_4} M_{\mathbb{Z}}^{(2,2)}$.

$$
\begin{aligned}
x_{aa}^\sharp &= \sum_{w \in \mathfrak{S}_4} (-1)^{\ell(w)} w \\
x_{bb}^\sharp &= (1) - (12) - (13) - (23) + (123) + (132) \\
x_{bc}^\sharp &= (34) - (12)(34) - (143) - (243) + (1243) + (1432) \\
x_{bd}^\sharp &= (234) - (1342) - (1423) - (24) + (13)(24) + (142) \\
x_{cc}^\sharp &= (1) - (12) - (14) - (24) + (124) + (142) \\
x_{cb}^\sharp &= (34) - (12)(34) - (134) - (234) + (1234) + (1342) \\
x_{cd}^\sharp &= (23) - (132) - (14)(23) - (243) + (1324) + (1432) \\
x_{db}^\sharp &= (243) - (1243) - (1324) - (24) + (124) + (13)(24) \\
x_{dc}^\sharp &= (23) - (123) - (14)(23) - (234) + (1234) + (1423) \\
x_{dd}^\sharp &= (1) - (13) - (14) - (34) + (134) + (143).
\end{aligned}
$$

It is easy to see that r does not belong to the R-linear span of these basis elements. This shows that $\dim_R \mathrm{Ann}_{R\mathfrak{S}_4} M_R^{(2,2)} \geqslant 11$. (In fact, it equals 11.)

8. Applications

We give some consequences of Theorem 7.3. As above, we work over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ in this section. First we note the following consequence of the main results.

Remark 8.1. As already noted in Section 3, we obtain the following result immediately by applying the involution $\sharp$ to the equalities in Theorems 5.2 and 7.3:

$$\mathrm{Ann}_{\mathscr{H}_R} \widetilde{M}_R^\lambda = \mathscr{H}_R[\not\trianglerighteq \lambda']$$

provided $\mathscr{H}_R$ is semisimple over a field R, or $R = \mathcal{A}$.

The next application is the cellularity of the algebras $\mathscr{H}_\mathcal{A}/(\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda)$.

Corollary 8.2. *Let $\lambda \vdash n$. The quotient algebra $\mathscr{H}_\mathcal{A}/(\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda)$ is cellular with cell basis $\{x_{st}^\sharp + \mathscr{H}_\mathcal{A}[\not\trianglerighteq \lambda']^\sharp : s, t \in \mathrm{Tab}(\mu), \mu \trianglelefteq \lambda'\}$.*

Proof. This follows immediately from the Theorem and the theory of cellular algebras. $\square$

Finally, we observe that certain other modules have the same annihilator as $M_\mathcal{A}^\lambda$. Recall that $\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]^\sharp/\mathscr{H}_\mathcal{A}[\triangleright\lambda]^\sharp$ is naturally a right $\mathscr{H}_\mathcal{A}$-module under right multiplication by elements of $\mathscr{H}_\mathcal{A}$ (see §3).

Corollary 8.3. *For any $\lambda \vdash n$, we have the equalities*

$$\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} \mathscr{H}_\mathcal{A}[\trianglerighteq\lambda] = \mathrm{Ann}_{\mathscr{H}_\mathcal{A}}(\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]/\mathscr{H}_\mathcal{A}[\triangleright\lambda])$$
$$= \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda = \mathscr{H}_\mathcal{A}[\not\trianglerighteq \lambda']^\sharp.$$

Proof. Recall that $M_\mathcal{A}^\lambda$ has a basis given by all $x_\lambda T_{d(t)}$ as t ranges over the set of row-standard tableaux of shape λ. Now, any $a \in \mathscr{H}_\mathcal{A}$ acting as zero on $x_\lambda T_{d(t)}$ also acts as zero on $T_{d(s)}^* x_\lambda T_{d(t)}$, for any s. Thus, if $a \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda$, then a acts as zero on all $x_{st} = T_{d(s)}^* x_\lambda T_{d(t)}$, with $s, t \in \mathrm{Tab}(\lambda)$. Since the images of these elements in $\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]/\mathscr{H}_\mathcal{A}[\triangleright\lambda]$ form a basis for that $\mathscr{H}_\mathcal{A}$-module, it follows that $a \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}}(A^x[\trianglerighteq\lambda]/A^x[\triangleright\lambda])$. This proves the inclusion $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda \subseteq \mathrm{Ann}_{\mathscr{H}_\mathcal{A}}(\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]/\mathscr{H}_\mathcal{A}[\triangleright\lambda])$. The opposite inclusion is clear, as $M_\mathcal{A}^\lambda$ is isomorphic with a submodule of $\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]/\mathscr{H}_\mathcal{A}[\triangleright\lambda]$. This proves the second equality.

By Lemma 7.1, we know that $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda \subseteq \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\mu$ for any $\lambda \trianglelefteq \mu$. Thus, by the result of the previous paragraph, it follows that any $a \in \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda$ acts as zero on all x_{uw} for (u, w) of shape μ, for any $\mu \trianglerighteq \lambda$. Since $\mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]$ is generated over $\mathcal{A}$ by such x_{uw}, it follows that $\mathrm{Ann}_{\mathscr{H}_\mathcal{A}} M_\mathcal{A}^\lambda \subseteq \mathrm{Ann}_{\mathscr{H}_\mathcal{A}} \mathscr{H}_\mathcal{A}[\trianglerighteq\lambda]$. The opposite inclusion is clear, so this proves the first equality. The third equality is from Theorem 7.3. $\square$

References

[1] R. Dipper and G. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* (3) **52** (1986), 20–52.

[2] R. Dipper and G. James, The q-Schur algebra, *Proc. London Math. Soc.* (3) **59** (1989), 23–50.

[3] R. Dipper and G. James, q-tensor space and q-Weyl modules, *Trans. Amer. Math. Soc.* (1) **327** (1991), 251–282.

[4] R. Dipper and G. James, Blocks and idempotents of Hecke algebras of general linear groups, *Proc. London Math. Soc.* (3) **54** (1987), 57–82.

[5] A. Giaquinto, Quantization of tensor representations and deformation of matrix bialgebras, *J. Pure Appl. Algebra*, **79** (1992), 169–190.

[6] J.J. Graham and G.I. Lehrer, Cellular algebras, *Invent. Math.* **123** (1996), 1–34.

[7] I. Grojnowski and G. Lusztig, On bases of irreducible representations of quantum GL_n, in "Kazhdan-Lusztig theory and related topics", *Contemp. Math.* **139** (1992), 167–174.

[8] M. Härterich, Murphy bases of generalized Temperley-Lieb algebras, *Archiv Math.* **72** (1999), 337–345.

[9] G.D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. **682**, Springer-Verlag, Berlin 1978.

[10] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.* **11** (1986), 247–252.

[11] A. Mathas, *Iwahori–Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, 15, American Mathematical Society, Providence, RI, 1999.

[12] G.E. Murphy, On the representation theory of the symmetric groups and associated Hecke algebras, *J. Algebra* **152** (1992), 492–513.

[13] G.E. Murphy, The representations of Hecke algebras of type A_n, *J. Algebra* **173** (1995), 97–121.

[14] N. Xi, Root vectors in quantum groups, *Comment. Math. Helv.* **69** (1994), 612–639.

Mathematics and Statistics, Loyola University Chicago, Chicago, Illinois 60626 U.S.A.

E-mail address: sdoty@luc.edu

Mathematics Department, Willamette University, 900 State Street, Salem, Oregon 97301 U.S.A.

E-mail address: knyman@willamette.edu